JOURNAL OF AIRCRAFT
Vol. 42, No. 2, March–April 2005

Engineering Notes

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Upper-Bound Flutter Speed Estimation Using the μ-k Method

Dan Borglund*
Royal Institute of Technology, SE-100 44 Stockholm, Sweden

Introduction

ROBUST flutter analysis deals with aeroelastic (or aeroservoelastic) stability analysis taking model uncertainty into account. Previous works have exclusively dealt with the computation of a worst-case flutter speed.[1–3] Although a lower bound on the flutter speed is of greater importance, an upper bound would also be useful in procedures for and evaluation of flight flutter testing. Although a lower bound provides a speed of flight from which a flutter instability can occur, an upper bound would provide a speed of flight that cannot be exceeded without encountering flutter. The complete robust analysis would thus divide the flight envelope into the three regions stable, stable or unstable, and unstable.

The quality of the robust flutter prediction is inherently dependent on the uncertainty description. This makes the development and validation testing of uncertainty descriptions for aeroservoelastic models very important. If the flutter speed obtained in flight or wind-tunnel testing does not fall within the predicted bounds, the uncertainty description might not capture the true source of uncertainty. An upper-bound estimation can thus be of significant value in future development of robust flutter analysis and flutter testing.[3,4]

This Note is an extension of Borglund,[3] showing how the μ-k method for robust flutter analysis can be used for an upper-bound flutter speed estimation. A simple model validation test based on the experimental flutter condition is also provided. Finally, the extended procedure for robust flutter analysis is found to give a more complete picture of the case study in Ref. 3.

The μ-k Method

Linear flutter analysis traditionally aims at solving the Laplace-domain flutter equation

$$\left[\boldsymbol{M}_0 p^2 + (L^2/V^2)\boldsymbol{K}_0 - (\rho L^2/2)\boldsymbol{Q}_0(p)\right]\boldsymbol{\eta} = \boldsymbol{F}_0(p)\boldsymbol{\eta} = \boldsymbol{0} \quad (1)$$

where V is the true airspeed; L is the aerodynamic reference length; ρ is the air density; $\boldsymbol{\eta}$ is the vector of generalized coordinates; $\boldsymbol{M}_0$ is the generalized mass matrix; $\boldsymbol{K}_0$ is the generalized stiffness matrix; and $\boldsymbol{Q}_0(p)$ is the generalized aerodynamic transfer matrix. Structural modal damping and the dependence on flight Mach number have been omitted for simplicity but can easily be taken into account. The nondimensional Laplace variable $p = g + ik$, where g is the the nondimensional damping, $k = \omega L/V$ is the reduced frequency, and ω is the frequency of the structural vibration.

The flutter equation (1) is a nonlinear eigenvalue problem that defines a set of eigenvalues p and corresponding eigenvectors $\boldsymbol{\eta}$ in the absence of model uncertainty. These eigenvalues will be referred to as the nominal eigenvalues. If all nominal eigenvalues have negative real parts, $g < 0$, the flight condition is nominally stable. The nominal flutter boundary is reached when some eigenvalue crosses the stability boundary $g = 0$ and can be computed using established methods for flutter analysis such as the p-k and g methods.[5,6]

The μ-k method is a straightforward frequency-domain procedure to compute flutter boundaries in the presence of model uncertainty.[3] The flutter equation (1) is modified to include uncertain dynamics and posed in the form

$$[\boldsymbol{I} - \boldsymbol{F}(p)\Delta(p)]\boldsymbol{z} = \boldsymbol{0} \quad (2)$$

defined by a flutter matrix $\boldsymbol{F}(p)$, a norm-bounded block-structured uncertainty matrix $\Delta(p)$, and a new set of variables $\boldsymbol{z}$. The matrix $\boldsymbol{F}(p)$ is completely determined by the flutter equation (1) and the uncertainty description and consequently depends on the flight condition.[3]

The uncertainty description typically represents perturbations to inertial, damping, elastic, and aerodynamic forces.[2,3] For example, a versatile uncertainty description for the aerodynamic forces was derived in Ref. 3 by assigning uncertainty to the lifting surface-pressure coefficients. If this uncertainty description is adopted, the uncertain aerodynamic matrix can be written as

$$\boldsymbol{Q}(p) = \boldsymbol{Q}_0(p) + \boldsymbol{V}_Q(p)\Delta_Q \boldsymbol{W}_Q \quad (3)$$

where $\boldsymbol{Q}_0(p)$ is the nominal aerodynamic matrix and Δ_Q is a diagonal uncertainty matrix with unknown complex parameters δ_j on the diagonal. In short, each parameter δ_j represents uncertainty in the pressure distribution for a certain region of the lifting surface. The weighting matrix $\boldsymbol{W}_Q$ determines the magnitude of the uncertainty such that each parameter belongs to the set $|\delta_j| \leq 1$ and $\boldsymbol{V}_Q(p)$ is an aerodynamic perturbation matrix that determines how the uncertain parameters influence the aerodynamic forces. By replacing $\boldsymbol{Q}_0(p)$ in Eq. (1) with $\boldsymbol{Q}(p)$ defined in Eq. (3), the matrices in Eq. (2) are easily identified as

$$\boldsymbol{F}(p) = (\rho L^2/2)\boldsymbol{W}_Q \boldsymbol{F}_0^{-1}(p)\boldsymbol{V}_Q(p) \quad (4)$$

$$\Delta(p) = \Delta_Q \quad (5)$$

along with the new variables $\boldsymbol{z} = \boldsymbol{W}_Q\boldsymbol{\eta}$ and where $\boldsymbol{F}_0(p)$ is the nominal flutter matrix defined in Eq. (1).

For a more general uncertainty description it is reasonable to assume that $\Delta(p)$ has the same properties (such as stability) as the transfer matrices being part of the flutter equation (1) (see Ref. 7). The uncertainty matrix $\Delta(p)$ is therefore assumed to belong to a set $\mathcal{S}_\Delta$ that is basically defined as

$$\mathcal{S}_\Delta = \{\Delta(p) : \Delta(p) \quad \text{stable and continuous for} \quad k > 0 \quad \text{and} \quad \Delta(p) \in \boldsymbol{\Delta} \quad \text{and} \quad \|\Delta(p)\|_\infty \leq 1\} \quad (6)$$

where the set $\boldsymbol{\Delta}$ defines the block structure $\|\Delta(p)\|_\infty = \sup_k \bar{\sigma}[\Delta(ik)]$, and $\bar{\sigma}[\Delta(ik)]$ is the maximum singular value of $\Delta(ik)$. The convenient norm bound $\|\Delta(p)\|_\infty \leq 1$ is achieved by scaling of the uncertainty set [3].

Received 15 January 2004; revision received 12 May 2004; accepted for publication 12 May 2004.

*Research Associate, Department of Aeronautical and Vehicle Engineering, Division of Flight Dynamics, Teknikringen 8. Member AIAA.

Flutter stability of the uncertain system is governed by the eigenvalues of Eq. (2). If all eigenvalues have negative real parts for all $\Delta(p) \in \mathcal{S}_\Delta$, the flight condition is robustly stable (stable in the presence of uncertainty). The μ-k method is a direct application of structured singular value analysis[8] to determine if some $\Delta(p) \in \mathcal{S}_\Delta$ enables a critical eigenvalue $p = ik$, making the corresponding matrix $\boldsymbol{I} - \boldsymbol{F}(ik)\Delta(ik)$ singular. The structured singular value μ of $\boldsymbol{F}(ik)$ is defined as the reciprocal of the minimum norm of any matrix $\Delta \in \boldsymbol{\Delta}$ making $\boldsymbol{I} - \boldsymbol{F}(ik)\Delta$ singular

$$\mu[\boldsymbol{F}(ik)] = 1 \Big/ \min_{\Delta \in \boldsymbol{\Delta}} \{\bar{\sigma}(\Delta) : \det[\boldsymbol{I} - \boldsymbol{F}(ik)\Delta] = 0\} \tag{7}$$

If no $\Delta \in \boldsymbol{\Delta}$ makes $\boldsymbol{I} - \boldsymbol{F}(ik)\Delta$ singular, $\mu[\boldsymbol{F}(ik)] = 0$. Now assume that

$$\mu(k) = \mu[\boldsymbol{F}(ik)] < 1 \tag{8}$$

holds for for all frequencies $k > 0$. Then no eigenvalue $p = ik$ is possible for $\Delta(p) \in \mathcal{S}_\Delta$. Because the eigenvalues of Eq. (2) are continuous functions of $\Delta(p)$ (Ref. 9), this also means that no nominal eigenvalue can be shifted to the stability boundary $g = 0$ by the uncertainty. Consequently, if the flight condition is nominally stable/unstable and $\mu(k) < 1$ for all $k > 0$, it is also robustly stable/unstable. The worst-case/best-case flutter boundary is thus characterized by nominally stable/unstable flight conditions for which the maximum value of $\mu(k) = 1$. This will be discussed in more detail in the next section.

Because only the frequency-domain aerodynamic forces are required to compute $\mu(k)$, established aerodynamic methods[10,11] can be used for the robust flutter analysis. However, the actual solution of the optimization problem in Eq. (7) is computationally very difficult. The robust stability criterion Eq. (8) is therefore evaluated using computable bounds for μ (Ref. 12). If the block structure $\boldsymbol{\Delta}$ is complex or mixed real/complex, useful upper and lower bounds for μ can be efficiently computed using established software packages.[13] In robust flutter analysis this is always the case if uncertainty in the aerodynamic forces is considered.[3]

Robust Flutter Analysis and μ-k Graphs

With the μ-k method, each flight condition is associated with a μ-k graph. When a nominal eigenvalue approaches the stability boundary $g = 0$, the minimum norm of the uncertainty required to destabilize the system tends to zero. In the μ-k graph this will become visible as a peak in the neighborhood of the frequency of the nominal mode. If the nominal mode is stable/unstable and the corresponding peak satisfies $\mu(k) < 1$, the mode is also robustly stable/unstable. The upper- and lower-bound flutter speeds for a certain mode can thus be determined by detecting when the corresponding peak crosses the robust stability boundary $\mu(k) = 1$.

Figure 1 illustrates the μ-k graphs for a case where a nominal eigenvalue moves from the stable region $g < 0$ into the unstable region $g > 0$ when the airspeed V is increased at constant flight altitude. The right-most graph corresponds to a low airspeed for which the system is robustly stable. When the airspeed increases to $V = V_{\text{low}}$, the maximum value of $\mu(k) = 1$, and V_{low} is the lower-bound (worst-case) flutter speed. For airspeeds higher than V_{low}, a flutter instability can occur in the frequency range $k \in [k_{\text{low}}, k_{\text{upp}}]$, where $\mu(k) \geq 1$. This range is defined by the flutter frequencies that the uncertain dynamics can produce at the corresponding flight condition. At the nominal flutter speed $V = V_{\text{nom}}$ the μ-k graph displays a singularity at the nominal flutter frequency $k = k_{\text{nom}}$ because no uncertainty is required to destabilize the system at this condition. If the airspeed is further increased, the nominal eigenvalue moves into the unstable region, and the peak eventually crosses the robust stability boundary $\mu(k) = 1$ at the upper bound (best-case) flutter speed $V = V_{\text{upp}}$.

If the peak of the critical mode is sufficiently distinct so that the crossing of the robust stability boundary $\mu(k) = 1$ can be detected, the μ-k method can thus be used to compute a lower (always possible) as well as an upper bound on the flutter speed. If the robust flutter analysis is successful, the experimental flutter speed V_{exp} obtained in flight or wind-tunnel testing should clearly satisfy

$$V_{\text{exp}} \in [V_{\text{low}}, V_{\text{upp}}] \tag{9}$$

If the flutter speed falls within the predicted bounds, the flutter frequency k_{exp} should in turn satisfy

$$k_{\text{exp}} \in [k_{\text{low}}, k_{\text{upp}}] \tag{10}$$

where the feasible frequency range $[k_{\text{low}}, k_{\text{upp}}]$ is defined by the μ-k graph for $V = V_{\text{exp}}$ (exemplified in Fig. 1). A substantial violation of Eqs. (9) or (10) would indicate a problem with the robust analysis, for example, that the uncertainty description does not capture the true uncertainty mechanism.

Table 1 Comparison between predicted and experimental flutter speeds and frequencies

Property	Nom.	Low.	Upp.	Exp.
V, m/s	14.7	13.3	16.4	16.0
f, Hz	6.47	6.35	6.42	6.40

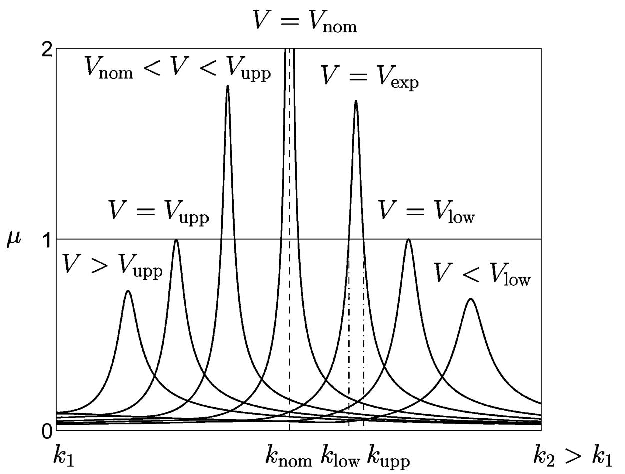

Fig. 1 Example μ-k graphs when the airspeed V is increased at constant altitude: horizontal ——, robust flutter boundary; vertical – – –, nominal flutter frequency; and vertical –.–.–, bounds on the reduced frequency range in which a flutter instability can occur.

Application to a Wind-Tunnel Model

In Ref. 3 a robust flutter analysis considering wing-tip aerodynamic uncertainty was developed in MATLAB® (Ref. 13) for a wind-tunnel model in low-speed airflow. The nominal analysis predicted flutter at $V_{\text{nom}} = 14.7$ m/s, the robust analysis provided the worst-case flutter speed $V_{\text{low}} = 13.3$ m/s, and wind-tunnel testing resulted in $V_{\text{exp}} = 16.0$ m/s. Because the experimental flutter speed was higher than the nominal flutter speed, no final conclusion on the quality of the robust flutter prediction could be made.

In Table 1 the results of a complete robust flutter analysis are compared to the experimental values in terms of flutter speed V and flutter frequency f. The μ-k graphs for the frequency range of interest are those illustrated in Fig. 1. Both the experimental flutter speed and the flutter frequency are found to be within the predicted bounds. This indicates that the aerodynamic uncertainty description developed in Ref. 3 is useful in practice and that a fairly accurate robust flutter prediction can be obtained. This insight would not have been possible without the upper-bound prediction.

Conclusions

This Note has shown that the μ-k method for robust flutter analysis can be used for an upper-bound flutter speed estimation. The flight envelope can then be divided into the three regions stable, stable or unstable, and unstable, which is particularly useful for model validation purposes in connection to flutter testing. The extended procedure for robust flutter analysis was successfully applied to a wind-tunnel model in low-speed airflow and was found to enable a more complete judgment on the usefulness of the aerodynamic uncertainty description and the accuracy of the robust flutter analysis.

Acknowledgments

This work was financially supported by The National Program for Aeronautics Research. The author thanks Ulf Ringertz for providing the implementation of the doublet-lattice method used in the case study.

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